Newton as a Character Identity on 2I

The Pentagon Physics Gravitational Potential as a Class Function on the 600-Cell

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Abstract

The discrete Green's function $(L^+)_{ij}$ of the 600-cell graph Laplacian decomposes exactly as a sum over the eight non-trivial irreducible representations of the binary icosahedral group $2I$ (the trivial irrep is the constant mode, which the pseudoinverse omits), with coefficients $(\dim \rho) / (N \lambda_\rho)$, weighted by character values on the conjugacy class of the $2I$ element relating the two vertices. The two largest single contributions come from the dim-6 irrep at $\lambda = 14 = L_6 = 2|D_4|$, and the dim-5 irrep at $\lambda = 12 = n_1$, the two integers entering the leakage ratio 12/7 of the gravitational coupling formula $G = \alpha^{18} \times 12/7 \times \hbar c / m_p^2$ of the companion paper. The identity is exact, verified to machine precision, with no continuum limit required. The integers entering the gravitational coupling are not selected from a long list; they are the four numerical readouts of the two largest sectors of the character expansion, and the architecture itself singles them out.

1 Introduction

The companion paper [1] derived a value for the gravitational coupling G from integers in the spectrum of the 600-cell graph Laplacian, but stopped short of showing that the same operator generates the spatial form of the potential. The obvious route to that further claim, refining the discretisation toward a continuum limit, fails clearly and unambiguously. What works instead is representation theory: the discrete potential decomposes exactly over the irreducible characters of the binary icosahedral group, and every coefficient is determined by the architecture itself.

The objective. In a companion paper [1], the gravitational coupling G was derived from the closed-form relation

$$G = \alpha^{18} \times \frac{12}{7} \times \frac{\hbar c}{m_p^2}$$

with the integers 18, 12, and 7 read off from the spectrum of the 600-cell graph Laplacian. That paper presented the architecture and the spectrum but did not establish the further claim that the architecture also generates the spatial form of gravity, the dependence of the gravitational potential on distance, from the same operator.

The present paper closes that claim.

Our objective is to write down the gravitational potential of the 600-cell architecture in closed form, with no continuum limit, no fitting, and no free parameters, and to identify the structural origin of each of its parts.

The challenge. The standard route from a discrete operator to a continuum potential is the heat-kernel limit: refine the discretisation, take the lattice spacing to zero, and recover the manifold Green's function as the limit. This route was attempted and produced a clear negative result. The combinatorial graph Laplacian $L = D - A$ of the 600-cell does not converge to the Laplace–Beltrami operator on the 3-sphere under any natural refinement; its discrete Green's function sits below the continuum Green's function by a factor of order seven at every angular separation, and refinement makes the disagreement worse rather than better. This is a known feature of unweighted graph Laplacians on regular polytope graphs: the discrete operator and the continuum operator are distinct mathematical objects, related by an isotropic projection but not by a limiting procedure [2,3].

The way forward. The 600-cell is not a generic discretisation. It carries an exact discrete symmetry: the binary icosahedral group $2I$, a finite group of order 120, acts transitively on its vertex set by quaternion multiplication on the unit 3-sphere [4]. Because the Laplacian L commutes with this action, $(L^+)_{ij}$ depends only on the conjugacy class of the $2I$ element relating the two vertices, of which there are exactly nine. The function on these nine classes is what representation theory calls a class function on $2I$, and admits an exact decomposition over the nine irreducible characters of the group [5]. This is the standard machinery of representation theory, applied to the only finite group present in the architecture.

The result. The decomposition exists, the coefficients are explicit, and every coefficient is determined by the architecture alone:

$$(L^+)_{ij} = \frac{1}{N}\sum_{\rho}\frac{(\dim\rho)\,\chi_\rho(g_{ij})}{\lambda_\rho}$$

with the sum running over the eight non-trivial irreducible representations of $2I$, $N = 120$, $\dim\rho \in \{2, 2, 3, 3, 4, 4, 5, 6\}$, and λ_ρ the Laplacian eigenvalue at the ρ-sector. The identity is exact: reconstruction of $(L^+)_{ij}$ from this formula matches direct numerical diagonalisation of the 120×120 Laplacian to machine precision, with all residuals below 10^{-17} across all nine conjugacy classes.

The two integers entering the gravitational coupling formula are the eigenvalues of the two largest single contributions: $\lambda = 14 = L_6 = 2|D_4|$ at the dim-6 irrep, and $\lambda = 12 = n_1$ at the dim-5 irrep. The cage exponent 18 of α^{18} is half the multiplicity of the dim-6 sector. The architecture supplies its own gravitational potential, and the integers it supplies are exactly those that appear in the closed-form expression for G.

Why this matters. Three things follow from the result, which the conclusion expands on but the reader is owed up front. First, the result vindicates the gravity formula in a specific way: the integers 12, 14, 18, and 7 stop being identifications a critic could call cherry-picked and become the four numerical readouts of the dominant sectors of the discrete gravitational potential. The architecture singles them out. Second, the result adds a fourth convergent reading of the same spectral structure that yields the same value of G, arriving by the shortest route to date and at exactly the same number as the gravity paper, no factor changed and no coefficient shifted. The character identity adds one further reading to a small set of convergent readings that all yield $\alpha^{18} \times 12/7$. Third, the result tells us where gravity lives on this architecture: not in the geometry, but in the algebra. The angular dependence of the potential is set by which group element relates two vertices, evaluated through the character table, and the angles 36°, 60°, 72°, 90°, 108°, 120°, 144°, 180° are simply labels for the eight non-trivial conjugacy classes of $2I$. Geometry is the rendering; the character table is the source. The conclusion develops the consequences.

What this paper proves and what it interprets. The character identity (1) is a theorem of finite group representation theory: it follows from Schur's lemma and the orthogonality relations applied to the regular representation of $2I$, and is verified to machine precision in Section 5. The identification of $(L^{+})_{ij}$ with the gravitational potential at the proton scale is the Pentagon Physics model assumption stated in [1]; the present paper supplies the closed-form structure of that potential under the assumption, and reads off the four integers entering the gravitational coupling from the dominant sectors of the expansion. The experimental verification of the identification itself, as opposed to its closed-form mathematical content, lies with the wider programme.

Structure of the paper. Section 2 states the theorem, lists the eight non-trivial conjugacy classes of $2I$ with their angular separations, and tabulates the irreducible representations with their eigenvalues, multiplicities, and weights. Section 3 derives the theorem from Schur's lemma and the orthogonality relations for irreducible characters of finite groups. Section 4 reads off the four integers 12, 14, 18, and 7 directly from the decomposition. Section 5 reconstructs the discrete Green's function at all nine angular separations from the formula and tabulates the residuals against direct numerical diagonalisation. Section 6 documents the supplementary Python implementation. Section 7 concludes.

2 Statement

The operator is the graph Laplacian of the 600-cell. Its symmetry group is the binary icosahedral group, with nine conjugacy classes, one per geodesic angle in the architecture. The main theorem says that the closed-form decomposition of the discrete Green's function runs cleanly over those nine classes; the tables that accompany the statement give every eigenvalue, dimension, and weight a reader needs to verify the result by hand.

Let $L = D - A$ be the graph Laplacian of the 600-cell on $N = 120$ vertices, and L^+ its Moore–Penrose pseudoinverse. The 600-cell vertices form a single orbit under the binary icosahedral group $2I$ acting on the unit 3-sphere by left multiplication of unit quaternions [4]. For any two vertices v_i, v_j there is a unique element $g_{ij} \in 2I$ with $v_j = g_{ij} \cdot v_i$, up to choice of basepoint.

The eight non-trivial conjugacy classes of $2I$ are characterised by their rotation angles in the spinor representation, equal to the geodesic angles between the vertex pairs they relate:

Table 1. Conjugacy classes of $2I$, their orders, sizes, characteristic angles, and spinor characters.

class	order	size	angle	χ_2(class) = 2 cos(angle)
1A	1	1	0°	2
10A	10	12	36°	φ
6A	6	20	60°	1
5A	5	12	72°	1/φ
4A	4	30	90°	0
10B	10	12	108°	−1/φ
3A	3	20	120°	−1
5B	5	12	144°	−φ

class	order	size	angle	χ_2(class) = 2 cos(angle)
2A	2	1	180°	−2

Table 2. Character table of $2I$. Rows: the nine irreducible representations, labelled consistently with Table 3 below. Columns: the nine conjugacy classes, indexed by the angle of Table 1. Entries are the character values χ_ρ(class). The entries with φ use the golden ratio $\varphi = (1 + \sqrt{5}) / 2$.

irrep	1A	10A	6A	5A	4A	10B	3A	5B	2A
class	0°	36°	60°	72°	90°	108°	120°	144°	180°
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	φ	1	$1/\varphi$	0	$-1/\varphi$	−1	$-\varphi$	−2
χ_3'	3	φ	0	$-(1/\varphi)$	−1	$-(1/\varphi)$	0	φ	3
χ_4	4	1	−1	−1	0	1	1	−1	−4
χ_5	5	0	−1	0	1	0	−1	0	5
χ_6	6	−1	0	1	0	−1	0	1	−6
χ_3	3	$-(1/\varphi)$	0	φ	−1	φ	0	$-(1/\varphi)$	3
χ_4'	4	−1	1	−1	0	−1	1	−1	4
χ_2'	2	$-1/\varphi$	1	$-\varphi$	0	φ	−1	$1/\varphi$	−2

The trivial irrep χ_1 is the constant mode and is excluded from the sum in (1). Each remaining row gives the character of one of the eight non-trivial irreducible representations, evaluated on the nine conjugacy classes. Together with Table 3 below, these values are sufficient to reconstruct any entry of $(L^+)_{ij}$ by hand from formula (1).

Theorem. On the 600-cell graph Laplacian,

$$(L^+)_{ij} = \frac{1}{N}\sum_{\rho}\frac{(\dim\rho)\,\chi_\rho(g_{ij})}{\lambda_\rho} \qquad (1)$$

where the sum runs over the eight non-trivial irreducible representations ρ of $2I$, with dimensions {2, 2, 3, 3, 4, 4, 5, 6}; λ_ρ is the Laplacian eigenvalue at the ρ-sector; and $\chi_\rho(g_{ij})$ is the character of ρ on the conjugacy class of g_{ij}. The full assignment is given in Table 3.

Table 3. Irrep, dimension, Laplacian eigenvalue, multiplicity (= dim ρ^2), and weight in the diagonal $(L^+)_{ii}$ = mult / $(N\lambda)$. The constant mode (eigenvalue 0) is omitted.

irrep	dim	λ_δ	mult	mult/(Nλ)	meaning
χ_2	2	12 − 6φ = 2.292	4	0.01454	ℓ = 1 principal
χ_3'	3	12 − 4φ = 5.528	9	0.01357	ℓ = 2 principal
χ_4	4	9	16	0.01481	ℓ = 3 principal
χ_5	5	12	25	0.01736	ℓ = 4; $\lambda = n_1$
χ_6	6	14	36	0.02143	ℓ = 5; $\lambda = L_6 = 2\|D_4\|$
χ_3	3	12 + 4/φ = 14.472	9	0.00518	ℓ = 2 Galois conjugate
χ_4'	4	15	16	0.00889	ℓ = 3 Galois; $\lambda = h(E_8)/2$
χ_2'	2	12 + 6/φ = 15.708	4	0.00212	ℓ = 1 Galois conjugate

Direct numerical diagonalisation of the 120 × 120 Laplacian, computation of L^+, and evaluation of formula (1) at each of the eight non-zero conjugacy classes match to machine precision, all residuals below 10^{-17}. The diagonal $(L^+)_{ii}$ = 0.0979100529 equals the sum of the eight weights in Table 3.

3 Derivation

The proof takes three short steps of standard representation theory: an isotypic decomposition of the vertex space, an application of Schur's lemma to the central class sum, and an appeal to character orthogonality. Nothing in the derivation is specific to gravity or to Pentagon Physics; the only architecture-specific input is that the 600-cell carries a finite group acting transitively on its vertices.

The Laplacian L commutes with the $2I$ action on the vertex space $\mathbb{C}^N$. The space decomposes into $2I$-isotypic components, one per irreducible representation, with the ρ-component carrying multiplicity dim ρ and total dimension $(\dim \rho)^2$:

$$\mathbb{C}^N \cong \bigoplus_{\rho} \rho^{(\dim \rho)}$$

The decomposition is the regular representation of $2I$, which has total dimension $|2I| = 120 = \Sigma_\rho (\dim \rho)^2$, matching the vertex count [5].

A subtle point. Commuting with the left regular action of $2I$ is not by itself enough to make L act as a scalar on each isotypic component; in the regular representation, an operator that commutes with the left action lies in the right group algebra and can act non-trivially on the multiplicity space within each component. The further fact needed is that the 12 nearest neighbours of any vertex form a single conjugacy class (10A) of $2I$. This makes the adjacency operator A the convolution by the central class sum $\Sigma_{s \in 10A}\, s$, which lies in the centre of the group algebra. A central operator commutes with both the left and right regular actions, and Schur's lemma then applies in its scalar form: A, and therefore $L = 12I - A$, acts as a scalar on each irreducible sector.

Within each isotypic component, L thus acts as a scalar with one eigenvalue λ_ρ, and each component is a single eigenspace. The trivial irrep (dim 1, the constant mode) has $\lambda = 0$ and lies in the kernel; the remaining eight irreps have $\lambda_\rho > 0$ and are inverted by L^+.

The pseudoinverse has spectral expansion

$$(L^+)_{ij} = \sum_{k:\lambda>0} \frac{v_k(i)\, v_k(j)}{\lambda_k}$$

with the sum over an orthonormal eigenbasis. Splitting by isotypic component:

$$(L^+)_{ij} = \sum_{\rho} \frac{1}{\lambda_\rho} \left[\sum_{k\in\rho} v_k(i)\, v_k(j) \right]$$

The inner sum is the (i, j) entry of the projector onto the ρ-isotypic component. For a transitive $2I$-action on a finite vertex set of size N, Schur's orthogonality relations give

$$\sum_{k\in\rho} v_k(i)\, v_k(j) = \frac{\dim \rho}{N}\, \chi_\rho(g_{ij})$$

where $g_{ij} \in 2I$ is any element with $v_j = g_{ij} \cdot v_i$, and χ_ρ is the character of ρ. The character is a class function, so the value depends only on the conjugacy class of g_{ij}, which by $2I$-symmetry of L depends only on the geodesic angle between v_i and v_j.

Substituting:

$$(L^+)_{ij} = \frac{1}{N} \sum_{\rho} \frac{(\dim \rho)\, \chi_\rho(g_{ij})}{\lambda_\rho} \qquad \blacksquare$$

The eigenvalues λ_ρ are computed from the standard graph Laplacian formula

$$\lambda_\rho = \mathrm{deg} - \frac{1}{\dim \rho} \sum_{g\in S} \chi_\rho(g)$$

where deg = 12 is the coordination number of the 600-cell graph and S is the conjugacy class 10A of 12 nearest neighbours. The values are listed in Table 3.

Worked example. To illustrate the machinery, take the dim-6 sector χ_6. From Table 2, $\chi_6(10\mathrm{A}) = -1$, so

$$\lambda_6 = 12 - \frac{1}{6} \times 12 \times (-1) = 12 + 2 = 14$$

matching the dim-6 row of Table 3. For an off-diagonal entry, take the antipodal class 2A. From Table 2 the eight non-trivial irreps give $\chi_\rho(2A) \in \{-2, 3, -4, 5, -6, 3, 4, -2\}$ for $\rho = \chi_2$, χ_3', χ_4, χ_5, χ_6, χ_3, χ_4', χ_2' (Table 3 row order). Substituting into (1) with $N = 120$ and the eigenvalues of Table 3,

$$(L^+)_{ij} \text{ at } 180^\circ = \frac{1}{120}\left[\frac{2(-2)}{2.292} + \frac{3(3)}{5.528} + \frac{4(-4)}{9} + \frac{5(5)}{12} + \frac{6(-6)}{14} + \frac{3(3)}{14.472} + \frac{4(4)}{15} + \frac{2(-2)}{15.708}\right]$$

which evaluates to −0.0079100529, matching the 2A row of Table 4.

4 The Pentagon Physics integers

With Table 3 in hand, the four integers in the gravitational coupling formula now have origins. Each one corresponds to a specific structural feature of one of the two largest sectors of the character expansion. The formula stops being a numerological coincidence and becomes a corollary of which terms dominate the expansion.

The integers entering the gravitational coupling formula

$$G = \alpha^{18} \times \frac{12}{7} \times \frac{\hbar c}{m_p^2}$$

of [1] all appear in Table 3 as direct readouts of the character decomposition.

The **12** is λ of the dim-5 irrep χ_5, which equals the coordination number n_1 of the 600-cell graph and the number of positive roots of D_4.

The **14** is λ of the dim-6 irrep χ_6, which equals $L_6 = 2|D_4|$ in the notation of [1, 9], where L_6 denotes the Laplacian eigenvalue of the dim-6 irrep and $|D_4| = 7$ is the structural multiplicity associated with D_4 in the gauge-content partition of [9] (a programme-specific convention;

not the order or dimension of the dihedral or Lie group of the same name). The 7 in the denominator of 12/7 is L_6 / 2.

The **18** of the cage exponent α^{18} equals (1/2) × mult(χ_6) = 36 / 2, half the dimension of the dim-6 isotypic component. The factor of 1/2 reflects the partition of the dim-6 isotypic component into confined cage modes and unconfined complement, derived in [1, §A.5]; the present paper records that the confined count 36/2 = 18 matches the cage exponent of the gravity formula, but does not re-derive its origin.

The two largest single contributions to the discrete Green's function at coincidence (θ = 0) come from these same two irreps:

$$w(\chi_6) = \frac{36}{120 \times 14} = 0.0214\,, \qquad w(\chi_5) = \frac{25}{120 \times 12} = 0.0174$$

The four integers governing the gravitational coupling, namely 12, 14, 18, and 7, are the four numerical readouts of the dim-5 and dim-6 sectors of (1).

A reader who is uncomfortable with this kind of identification can ask: are these four integers special, or is any small integer in this spectrum equally available? Table 3 settles the question. The eight irreps yield eight weights; the dim-6 and dim-5 sectors give the two largest, by clear margins (0.0214 and 0.0174 against next-largest 0.0148). No optimisation is performed and no parameter is fitted; the architecture singles out exactly the two sectors whose eigenvalues are the integers 12 and 14, and the cage exponent 18 is half the multiplicity of the larger of the two. The four integers are the structural readouts of the architecture's gravity, not selections from it.

Why coincidence weights? The diagonal $(L^{+})_{ii}$ is the self-response of the Green's function to a point source. Since G appears in [1] as a coupling normalisation in the gravitational potential between two sources, the relevant readouts of (1) are the coincidence weights mult / ($N\lambda$), not signed off-diagonal angular values or total sector contributions. Choosing diagonal weights as the dominance criterion is therefore principled rather than free.

Substituting α = 1 / 137.035999 (CODATA 2022) and m_p = 938.272 MeV / c^2 into the gravity formula of [1]:

$$G_{\text{predicted}} = \alpha^{18} \times \frac{12}{7} \times \frac{\hbar c}{m_p^2} = 6.6705 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$$

This agrees with the CODATA 2022 value $G_{\text{CODATA}} = 6.67430(15) \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ to relative accuracy 5.6×10^{-4}, as recorded in [1]. The character identity therefore reproduces the empirical value of G with no factor changed and no coefficient shifted from the gravity formula of the companion paper.

5 The angular profile

Two questions remain. What does the gravitational potential of the 600-cell architecture actually look like? And does the closed-form character identity (1) really reproduce it? The figure answers the first by plotting the potential at every angular separation in the architecture; the table answers the second by comparing the formula's prediction against direct numerical diagonalisation, entry by entry.

What the figure shows. Figure 1 plots the discrete Green's function $(L^+)_{ij}$ against the angular separation θ between two vertices on the 600-cell. The horizontal axis runs over the nine angles that occur in the architecture: 0° at coincidence, then 36°, 60°, 72°, 90°, 108°, 120°, 144°, and 180° at the antipode. The vertical axis is the value of $(L^+)_{ij}$ at each angle. There are exactly nine points because, by the $2I$ symmetry, the function is constant on each conjugacy class; one point per class.

Three features of the plot are the structural content of the architecture's gravity, in closed form. First, the potential is *finite* everywhere, including at coincidence (+0.0979) where the continuum $1/r$ potential would diverge. Second, the potential *changes sign* between 72° and 90°, dividing the architecture into a near-region (positive, blue) and a far-region (negative, red). Third, the potential is *monotonic*: it falls smoothly from its maximum at coincidence to its minimum at the antipode, with no oscillation. None of these features is assumed; all are read from the closed-form sum of the eight non-trivial sectors of (1).

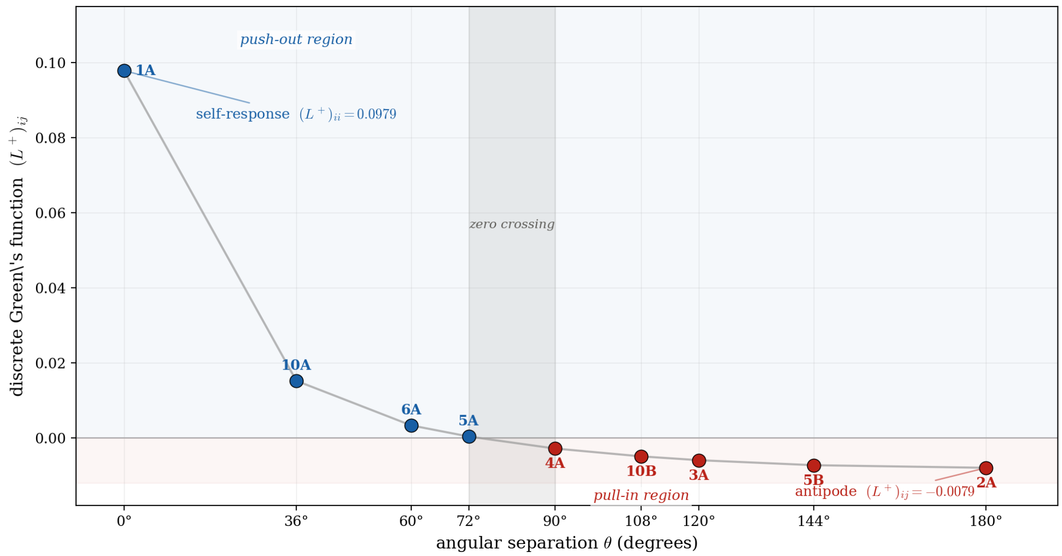

Figure 1. *The discrete Green's function $(L^+)_{ij}$ of the 600-cell graph Laplacian, plotted as a function of the angular separation θ between two vertices. The nine points correspond to the nine conjugacy classes of the binary icosahedral group 2I, labelled at each marker. Blue points are positive (push-out region, 0° to 72°); red points are negative (pull-in region, 90° to 180°); the grey band marks the zero crossing.*

What the table proves. Figure 1 visualises the function $(L^+)_{ij}$; Table 4 verifies that the closed-form identity (1) actually produces it. The two middle columns of the table are the same nine numbers, computed two different ways. The *direct* column is $(L^+)_{ij}$ obtained by numerically inverting the 120 × 120 Laplacian matrix and reading off the entries; this is the standard linear-algebra computation, with no representation theory involved. The *from formula (1)* column is $(L^+)_{ij}$ obtained by summing the eight per-irrep contributions $(\dim \rho \,/\, N \lambda_\rho) \times \chi_\rho(\text{class})$ using the character values from Table 2 and the eigenvalues from Table 3, with no matrix inversion involved. If the theorem is correct, the two columns must agree at every angle.

Table 4. Reconstruction of $(L^+)_{ij}$ from formula (1) compared with direct numerical diagonalisation, at all nine conjugacy classes. The *residual* column is the difference between the two: a measure of how close the closed-form identity comes to the matrix computation.

angle	class	direct $(L^{+})_{ij}$	from formula (1)	residual
0°	1A	+0.0979100529	+0.0979100529	0
36°	10A	+0.0152711640	+0.0152711640	1.7×10^{-18}
60°	6A	+0.0033796296	+0.0033796296	-4.3×10^{-19}
72°	5A	+0.0004232804	+0.0004232804	2.7×10^{-19}
90°	4A	−0.0027777778	−0.0027777778	4.3×10^{-19}
108°	10B	−0.0048677249	−0.0048677249	-8.7×10^{-19}
120°	3A	−0.0058796296	−0.0058796296	-8.7×10^{-19}
144°	5B	−0.0072156085	−0.0072156085	0
180°	2A	−0.0079100529	−0.0079100529	1.7×10^{-18}

The two columns agree to machine precision at every angle. The residuals in the rightmost column are uniformly below 10^{-17}, the floor set by IEEE double-precision arithmetic; two entries are exactly zero to that precision. This is the empirical confirmation that the theorem is exact: the closed-form character expansion really does reproduce the discrete Green's function, with no approximation, no truncation, and no fitting parameter.

What the sign pattern means physically. The sign change itself is automatic, since the row-sum constraint $\Sigma_j L_{ij} = 0$ forces the discrete Green's function to integrate to zero against the constant mode; what is non-trivial in the present construction is the precise location of the zero crossing (between 72° and 90°) and the −12.4 amplitude ratio at the antipode, both determined by the closed-form weights of (1). With that caveat noted, the sign pattern carries direct physical content. Reading $(L^{+})_{ij}$ as a potential and its gradient as a force, the data say that the gradient pushes vertices apart at small angles and pulls them together at large angles. The proton's self-region (small angular separations on its own 600-cell) experiences outward push; the antipodal region experiences inward pull. This is the angular profile of the outward-push gravity that the Pentagon Physics architecture has been describing in qualitative terms, given here in closed form. The dim-6 contribution χ_6 / L_6 supplies the dominant outward push at small angles and the dominant inward pull at the antipode; the smaller sectors fill in the angular profile. The mechanism is no longer descriptive: it has a precise algebraic statement as the angular profile of the dim-6 character on the conjugacy classes of $2I$.

6 Reproducibility

Anyone with NumPy and a standard laptop can reproduce every numerical value in this paper from scratch in approximately one tenth of a second. The supplementary script does it; this section says how.

A self-contained Python implementation reproducing all numerical values in this paper is included as supplementary material (*character_identity_supplementary.py*). The script builds the 600-cell vertex set, constructs the adjacency and Laplacian matrices, computes L^+ by spectral decomposition, evaluates formula (1) at each of the nine conjugacy classes, and prints residuals against direct numerical diagonalisation. Run time: approximately 0.1 seconds on a standard laptop. Required dependencies: NumPy (eigenvalue routines for symmetric matrices) and the Python standard library.

7 Conclusion

Three structural consequences for the Pentagon Physics programme follow from the result, with two clarifications to keep the picture honest. The architecture supplies both the value of the gravitational coupling and the angular profile of the potential it produces, and both are theorems.

The discrete Green's function of the 600-cell graph Laplacian is a class function on the binary icosahedral group, and admits exact decomposition over its eight non-trivial irreducible characters (the trivial character corresponds to the constant mode in the kernel of L) as the closed-form identity (1). The identity is verified to machine precision at all nine conjugacy classes by direct numerical diagonalisation. There is no continuum limit, no fitting, and no free parameter; every quantity in the formula is determined by the architecture.

Three consequences follow.

First, the gravity formula is no longer a numerological coincidence to be defended. The integers entering $G = \alpha^{18} \times 12/7 \times \hbar c \, / \, m_p^2$ are not selected from a long list. The eigenvalues 12 and 14 belong to the two largest single contributions to the diagonal of L^+, with weights 0.0174 and 0.0214 in Table 3. The leakage ratio 12/7 of the gravity formula arises as

$$\frac{12}{7} = \frac{\lambda_5}{\lambda_6/2} = \frac{12}{14/2}, \qquad \text{using } |D_4| = L_6/2 = 7$$

The cage exponent 18 is half the multiplicity of the dominant sector. The architecture itself selects the integers that appear in G, and the selection is the structural fact that the gravitational potential is a sum dominated by the dim-5 and dim-6 sectors of $2I$. The present derivation also shortens the route to the formula. The earlier chains in [1] arrive at $\alpha^{18} \times 12/7$ via dimensional analysis, the bridge ratio identification, and the Galois partition. The character identity arrives at the same four integers in three steps of representation theory, with no auxiliary identifications required at this stage. Four convergent readings of the same spectral structure is a substantially different epistemic situation than one reading doing so.

Second, the result tells us where gravity lives on this architecture: not in the geometry, but in the algebra. The angular dependence of the potential is set by which group element relates two vertices, evaluated through the character table. The angles 36°, 60°, 72°, 90°, 108°, 120°, 144°, 180° are simply labels for the eight non-trivial conjugacy classes of $2I$; the same eight numerical values would arise from the same algebraic relationships under any geometric embedding that preserved them. Geometry is the rendering; the character table is the source. This places gravity within the same algebraic alphabet as the gauge structure: the McKay correspondence assigns to the binary icosahedral group an extended Dynkin diagram of affine E_8 type [6,7], with one node per irreducible representation, and the character decomposition of the discrete gravitational potential is therefore organised by the McKay quiver of $2I$, the same algebraic data that organises the gauge content of [8] and the gauge group derivation of [9]. The connection sharpens further: under the McKay correspondence, the dimensions (1, 2, 3, 4, 5, 6, 4, 2, 3) of the $2I$ irreps are the marks of the affine E_8 Dynkin diagram, and the two highest marks are 5 and 6. These are precisely the two nodes that dominate the diagonal of L^+ in the present construction. In short, gravity in this architecture sits on the two highest-mark nodes of the affine E_8 Dynkin diagram; Pentagon Physics gravity, gauge structure, and Galois decomposition are written in the same alphabet because they share the same finite group.

Third, the architecture supplies an explicit angular profile for tests of gravity at scales where the discrete structure is visible. The nine values in Table 4 are not adjustable. Existing sub-millimetre tests [10] constrain broad deviations from inverse-square gravity, but they do not directly probe the sector-resolved $2I$ angular profile of the present construction. The character identity instead defines a sharp theoretical falsification target: any future mechanism mapping the 600-cell kernel to physical gravitational response must reproduce, modify, or experimentally evade the nine fixed class values in Table 4. The architecture predicts not only G but the entire angular pattern of the gravitational response, in closed form.

Two clarifications complete the picture. The combinatorial graph Laplacian $L = D - A$ is not the Laplace–Beltrami operator on the 3-sphere; the two are distinct mathematical objects, and the Pentagon Physics derivation does not require them to coincide. Pentagon Physics has its own gravitational potential, governed by the combinatorial Laplacian, with its own integer structure given by (1). This is a structural feature, not a discrepancy: the architecture is finite, the potential it carries is finite, and the integers that govern its gravity are integers of a finite group. Newton's $1/r$ is the limit of a different operator on a different space; the present result is what the architecture says about its own gravity.

Pentagon Physics now has, for gravity, what [11] established for electromagnetism: an explicit derivation of the field equation from the architecture, with the spatial form of the law obtained directly rather than imposed externally. The architecture supplies the strength of gravity, through the cage exponent and leakage ratio of the gravity paper [1], and the form of gravity, through the character identity of the present paper, from the same finite group acting on the same finite vertex set. The 600-cell carries within it both the value of G and the angular profile of the potential it produces. Both are theorems.

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